

Domination polynomial of generalized friendship and generalized book graphs

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ABSTRACT

Let G be a simple graph of order n . The domination polynomial of G is the polynomial $D(G, x) = \sum_{i=\gamma(G)}^n d(G, i)x^i$, where $d(G, i)$ is the number of dominating sets of G of size i and $\gamma(G)$ is the domination number of G . Let n be any positive integer and F_n be the *Friendship graph* with $2n + 1$ vertices and $3n$ edges, formed by the join of K_1 with nK_2 . We study the domination polynomials of generalized friendship graphs. We also consider the *n-book graphs* B_n , formed by joining n copies of the cycle graph C_4 with a common edge and study the domination polynomials of some generalized book graphs. In particular we examine the domination roots of these families, and find the limiting curve for the roots.

Mathematics Subject Classification: 05C60.

Keywords: Domination polynomial; friendship graph; flower graphs; generalized book graphs; family.

1 Introduction

Let $G = (V, E)$ be a simple graph. For any vertex $v \in V(G)$, the *open neighborhood* of v is the set $N(v) = \{u \in V(G) | \{u, v\} \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V(G)$, the open neighborhood of S is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. A set $S \subseteq V(G)$ is a *dominating set* if $N[S] = V$ or equivalently, every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S . The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G . For a detailed treatment of these parameters, the reader is referred to [10]. The *i-subset* of $V(G)$ is a subset of $V(G)$ of size i . Let $\mathcal{D}(G, i)$ be the family of dominating sets of a graph G with

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cardinality i and let $d(G, i) = |\mathcal{D}(G, i)|$. The *domination polynomial* $D(G, x)$ of G is defined as $D(G, x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G, i)x^i$, where $\gamma(G)$ is the domination number of G (see [1, 3]). A root of $D(G, x)$ is called a domination root of G . The set of distinct roots of $D(G, x)$ is denoted by $Z(D(G, x))$.

Calculating the domination polynomial of a graph G is difficult in general, as the smallest power of a non-zero term is the domination number $\gamma(G)$ of the graph, and determining whether $\gamma(G) \leq k$ is known to be NP-complete [9]. But for certain classes of graphs, we can find a closed form expression for the domination polynomial. In [4] the domination polynomial and the domination roots of friendship graphs has been studied. In this paper we would like to obtain some further results of this kind. We consider generalized friendship graph (or flower graphs), and generalized book graphs and calculate their domination polynomials. Also we explore the nature and location of their roots.

2 Domination polynomial of generalized friendship graphs

Let consider the graphs F_n obtained by selecting one vertex in each of n triangles and identifying them (Figure 1). Some call them Dutch-Windmill graphs [15] and friendship graphs.

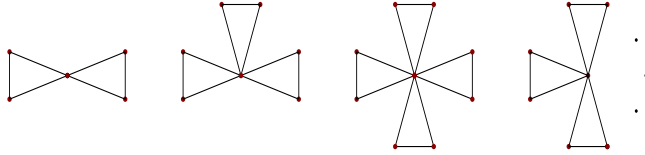


Figure 1: Friendship graphs F_2, F_3, F_4 and F_n , respectively.

The generalized friendship graph $F_{q,n}$ is a collection of n cycles (all of order q), meeting at a common vertex (see Figure 2). The generalized friendship graph may also be referred to as a flower [13].

In this section we compute the domination polynomial of the flowers $F_{4,n}$. We need some preliminaries.

The vertex contraction G/u of a graph G by a vertex u is the operation under which all vertices

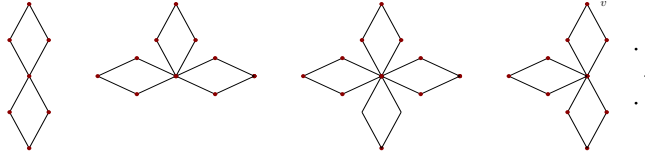


Figure 2: The flowers $F_{4,2}$, $F_{4,3}$, $F_{4,4}$ and $F_{4,n}$, respectively.

in $N(u)$ are joined to each other and then u is deleted (see[14]).

The following theorem is useful for finding the recurrence relations for the domination polynomials of graphs.

Theorem 1.[2, 11] *Let G be a graph. For any vertex u in G we have*

$$D(G, x) = xD(G/u, x) + D(G - u, x) + xD(G - N[u], x) - (1 + x)p_u(G, x),$$

where $p_u(G, x)$ is the polynomial counting the dominating sets of $G - u$ which do not contain any vertex of $N(u)$ in G .

The following theorem gives formula for the domination polynomial of F_n .

Theorem 2. [4] *For every $n \in \mathbb{N}$,*

$$D(F_n, x) = (2x + x^2)^n + x(1 + x)^{2n}.$$

Domination polynomial satisfies a recurrence relation for arbitrary graphs which is based on the edge and vertex elimination operations. The recurrence uses composite operations, e.g. $G - e/u$, which stands for $(G - e)/u$.

Theorem 3.[11] *Let G be a graph. For every edge $e = \{u, v\} \in E$,*

$$\begin{aligned} D(G, x) &= D(G - e, x) + \frac{x}{x - 1} \left[D(G - e/u, x) + D(G - e/v, x) \right. \\ &\quad - D(G/u, x) - D(G/v, x) - D(G - N[u], x) - D(G - N[v], x) \\ &\quad \left. + D(G - e - N[u], x) + D(G - e - N[v], x) \right]. \end{aligned}$$

The following theorem gives recurrence relation for the domination polynomial of $F_{4,n}$.

Theorem 4. *For every $n \geq 2$,*

$$\begin{aligned} D(F_{4,n}, x) &= ((1+x)^3 + x)D(F_{4,n-1}, x) - (1+3x)(x+3x^2+x^3)^{n-1} \\ &\quad + (1+x)^3x^{n-1} - (x^2+x)(x^3+3x^2+3x)^{n-1}, \end{aligned}$$

where $D(F_{4,1}, x) = x^4 + 4x^3 + 6x^2$.

Proof. An elementary observation is that if G_1 and G_2 are graphs of orders n_1 and n_2 , respectively, then

$$D(G_1 \cup G_2, x) = D(G_1, x)D(G_2, x).$$

Consider graph $F_{4,n}$ and a vertex v in Figure 2. By Theorem 1 we have:

$$\begin{aligned} D(F_{4,n}, x) &= xD(F_{4,n}/v, x) + D(F_{4,n} - v, x) + xD(F_{4,n} - N[v], x) - (1+x)p_v(F_{4,n}, x) \\ &= xD(F_{4,n}/v, x) + D(F_{4,n} - v, x) + xD(F_{4,n-1}, x) \\ &\quad - (1+x)x D(\cup_{i=1}^{n-1} K_3, x), \end{aligned} \tag{1}$$

where $D(K_3, x) = x^3 + 3x^2 + 3x$.

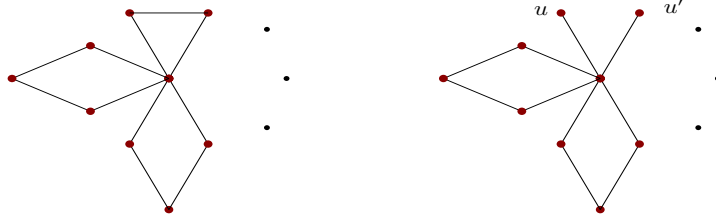


Figure 3: Graphs $F_{4,n}/v$ and $F_{4,n} - v$, respectively.

Now we use Theorem 3 to obtain the domination polynomial of the graph $F_{4,n}/v = G$ (see Figure 3). We have

$$\begin{aligned} D(F_{4,n}/v, x) &= D(F_{4,n} - v, x) + \frac{x}{x-1} \left[-D(\cup_{i=1}^{n-1} P_3, x) - D(\cup_{i=1}^{n-1} P_3, x) \right. \\ &\quad \left. + xD(\cup_{i=1}^{n-1} P_3, x) + xD(\cup_{i=1}^{n-1} P_3, x) \right] \\ &= D(F_{4,n} - v, x) + 2xD(P_3, x)^{n-1}, \end{aligned} \tag{2}$$

where $D(G - e/u, x) \simeq D(G - e/u', x) \simeq D(G/u, x) \simeq D(G/u', x)$ and $(F_{4,n}/v) - e \simeq F_{4,n} - v$ in Figure 3.

Now we use Theorem 1 to obtain the domination polynomial of the graph $F_{4,n} - v = G$ (see Figure 3). We have

$$\begin{aligned}
D(F_{4,n} - v, x) &= xD((F_{4,n} - v)/u, x) + D(F_{4,n} - v - u, x) + xD(F_{4,n} - v - N[u], x) \\
&\quad - (1 + x)p_u(F_{4,n} - v, x) \\
&= xD((F_{4,n} - v)/u, x) + D(F_{4,n} - v - u, x) + x(xD(\cup_{i=1}^{n-1} P_3, x)) \\
&\quad - (1 + x)xD(\cup_{i=1}^{n-1} P_3, x) \\
&= (1 + x)D((F_{4,n} - v)/u, x) - xD(P_3, x)^{n-1}.
\end{aligned} \tag{3}$$

Note that we used $D((F_{4,n} - v)/u, x) \simeq D(F_{4,n} - v - u, x)$ (see Figure 4).

Use Theorem 1 to obtain the domination polynomial of the graph $(F_{4,n} - v)/u = G$ (see Figure 4).

We have

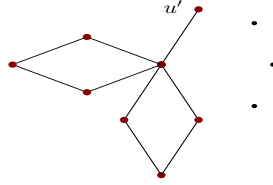


Figure 4: The graph $(F_{4,n} - v)/u$.

$$\begin{aligned}
D(F_{4,n} - v)/u, x) &= xD((F_{4,n} - v)/u', x) + D(F_{4,n} - v - u', x) + xD(F_{4,n} - v - N[u'], x) \\
&\quad - (1 + x)p_{u'}(F_{4,n} - v, x) \\
&= xD(F_{4,n-1}, x) + D(F_{4,n-1}, x) + xD(\cup_{i=1}^{n-1} P_3, x) \\
&\quad - (1 + x)(D(\cup_{i=1}^{n-1} P_3, x) - x^{n-1}) \\
&= (1 + x)D(F_{4,n-1}, x) - D(P_3, x)^{n-1} + (1 + x)(x^{n-1}),
\end{aligned} \tag{4}$$

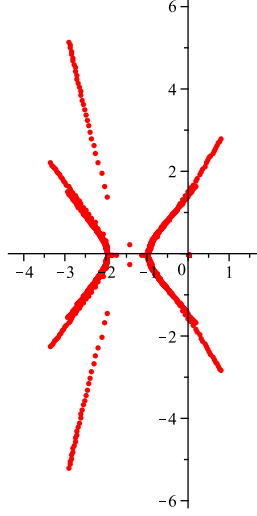


Figure 5: Domination roots of graphs $F_{4,n}$, for $1 \leq n \leq 30$.

Consequently, by equations 1, 2, 3 and 4 have:

$$\begin{aligned} D(F_{4,n}, x) &= (1 + 4x + 3x^2 + x^3)D(F_{4,n-1}, x) - (1 + 3x)(x + 3x^2 + x^3)^{n-1} \\ &\quad + (1 + x)^3 x^{n-1} - (x^2 + x)(x^3 + 3x^2 + 3x)^{n-1}. \quad \square \end{aligned}$$

The domination roots of $F_{4,n}$ exhibit a number of interesting properties (see Figure 5).

If we can find an explicit formula for the domination polynomial of a graph, there are still interesting, difficult problems concerning the roots. For every odd natural number n , no nonzero real number is a domination root of F_n [4]. Also we think that for n even, F_n have exactly three real roots. Using Maple, we think that these are true for $F_{4,n}$. Therefore, we pose the following:

Question 1 *For even $n \geq 4$, does F_n have exactly three real roots?*

Conjecture 1 *For every odd natural number n , no nonzero real number is a domination root of $F_{4,n}$.*

Question 2 *What is a good upper bound on the modulus of the roots of $F_{4,n}$?*

It is natural to ask about the complex domination roots of $F_{4,n}$. The plot in Figure 5 suggests that the roots tend to lie on a curve.

Conjecture 2 *The limit of domination roots of $F_{4,n}$ is hyperbola.*

In [7] a family of graphs was produced with roots just barely in the right-half plane (showing that not all domination polynomials are stable), but Figure 5 provides an explicit family (namely the $F_{4,n}$) whose domination roots have unbounded positive real part.

3 Domination polynomial of generalized book graphs

A book graph B_n , is defined as follows $V(B_n) = \{u_1, u_2\} \cup \{v_i, w_i : 1 \leq i \leq n\}$ and $E(B_n) = \{u_1 u_2\} \cup \{u_1 v_i, u_2 w_i, v_i w_i : 1 \leq i \leq n\}$. We consider the generalized book graph $B_{n,m}$ with vertex and edge sets by $V(B_{n,m}) = \{u_i : 1 \leq i \leq m-2\} \cup \{v_i, w_i : 1 \leq i \leq n\}$ and $E(B_{n,m}) = \{u_i u_{i+1} : 1 \leq i \leq m-3\} \cup \{u_i w_j : 1 \leq j \leq n, i = m-2\} \cup \{u_1 v_i : 1 \leq i \leq n\} \cup \{v_i w_i : 1 \leq i \leq n\}$ (see Figure 6).

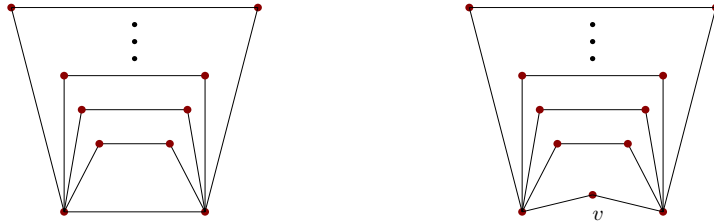


Figure 6: Graphs B_n and $B_{n,5}$, respectively.

The following theorem gives formula for the domination polynomial of B_n .

Theorem 5.^[4] *For every $n \in \mathbb{N}$,*

$$D(B_n, x) = (x^2 + 2x)^n(2x + 1) + x^2(x + 1)^{2n} - 2x^n.$$

Figure 7 shows the domination roots of book graphs B_n for $n \leq 30$.

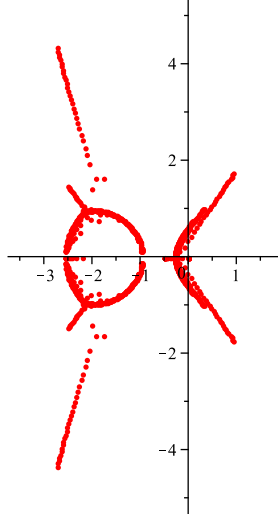


Figure 7: Domination roots of graphs B_n , for $1 \leq n \leq 30$.

In this section we compute domination polynomial of the book graphs $B_{n,5}$. We need some preliminaries.

We begin with a graph operation. For two graphs $G = (V, E)$ and $H = (W, F)$, the corona $G \circ H$ is the graph arising from the disjoint union of G with $|V|$ copies of H , by adding edges between the i th vertex of G and all vertices of i th copy of H [8]. It is easy to see that the corona operation of two graphs does not have the commutative property. The following theorem gives the domination polynomial of graphs of the form $H \circ K_1$, which is needed to obtain our result.

Theorem 6.[1] *Let G be a graph. Then $D(G, x) = (x^2 + 2x)^n$ if and only if $G = H \circ K_1$ for some graph H of order n .*

Given any two graphs G and H we define the *Cartesian product*, denoted $G \square H$, to be the graph with vertex set $V(G) \times V(H)$ and edges between two vertices (u_1, v_1) and (u_2, v_2) if and only if either $u_1 = u_2$ and $v_1 v_2 \in E(H)$ or $u_1 u_2 \in E(G)$ and $v_1 = v_2$. This product is well known to be commutative.

Theorem 7.[12] *The domination polynomial for $K_n \square K_2$ is*

$$D(K_n \square K_2, x) = ((1 + x)^n - 1)^2 + 2x^n.$$

The following theorem gives formula for the domination polynomial of $B_{n,5}$.

Theorem 8. *For every $n \in \mathbb{N}$,*

$$D(B_{n,5}, x) = x^2(x+1)^{2n+1} - 2x^{n+1} + (x^2 + 2x)^n(2x^2 + 3x).$$

Proof. Consider graph $B_{n,5}$ in Figure 6. By Theorems 1 and 5 we have:

$$\begin{aligned} D(B_{n,5}, x) &= xD(B_{n,5}/v, x) + D(B_{n,5} - v, x) + xD(B_{n,5} - N[v], x) \\ &\quad - (1+x)p_v(B_{n,5}, x) \\ &= xD(B_n, x) + D(B_{n,5} - v, x) + x(D(\cup_{i=1}^n K_2, x)) \\ &\quad - (1+x)[(x^2 + 2x)^n - 2x^n] \\ &= x[(x^2 + 2x)^n(2x + 1) + x^2(x+1)^{2n} - 2x^n] + D(B_{n,5} - v, x) \\ &\quad + x(x^2 + 2x)^n - (1+x)[(x^2 + 2x)^n - 2x^n] \\ &= x^3(x+1)^{2n} + 2x^n + D(B_{n,5} - v, x) + (x^2 + 2x)^n(2x^2 + x - 1). \end{aligned} \quad (5)$$

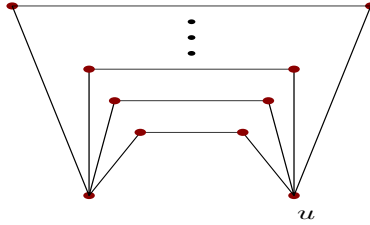


Figure 8: The graph $B_{n,5} - v$.

Now we use Theorem 1 to obtain the domination polynomial of the graph $B_{n,5} - v = G$ (see Figure 8). We have

$$\begin{aligned} D(B_{n,5} - v, x) &= xD(G/u, x) + D(G - u, x) + xD(G - N[u], x) - (1+x)p_u(G, x) \\ &= xD(G/u, x) + D(G - u, x) + xD(K_{1,n}, x) - (1+x)(x^n(1+x)) \\ &= xD(G/u, x) + D(G - u, x) + x(x^n + x(1+x)^n) - x^n(1+x)^2, \end{aligned} \quad (6)$$

where $D(K_{1,n}, x) = x^n + x(1+x)^n$.

Use Theorems 1 and 6 to obtain the domination polynomial of the graph $B_{n,5} - v - u = G$ (see Figure 9). We have

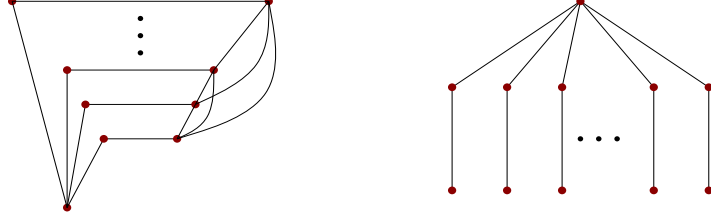


Figure 9: Graphs $(B_{n,5} - v)/u$ and $B_{n,5} - v - u$, respectively.

$$\begin{aligned}
D(B_{n,5} - v - u, x) &= xD(G/w, x) + D(G - w, x) + xD(G - N[w], x) \\
&\quad - (1 + x)p_w(G, x) \\
&= xD(K_n \circ K_1, x) + D(\cup_{i=1}^n P_2, x) + x(x^n) - (1 + x)x^n \\
&= (2x + x^2)^n(x + 1) - x^n.
\end{aligned} \tag{7}$$

Use Theorems 1 and 6 and 7 to obtain the domination polynomial of the graph $(B_{n,5} - v)/u = G$ (see Figure 9). We have

$$\begin{aligned}
D((B_{n,5} - v)/u, x) &= xD(G/u', x) + D(G - u', x) + xD(G - N[u'], x) - (1 + x)p_{u'}(G, x) \\
&= xD(K_n \square K_2, x) + D(K_n \circ K_1, x) + xD(K_n, x) - (1 + x)x^n \\
&= x((1 + x)^n - 1)^2 + 2x^n + (2x + x^2)^n + x((1 + x)^n - 1) - (1 + x)x^n \\
&= x((1 + x)^n - 1)(1 + x)^n + (2x + x^2)^n + x^n(x - 1).
\end{aligned} \tag{8}$$

By equations 6, 7 and 8 have:

$$D(B_{n,5} - v, x) = (x^2 + 2x)^n(2x + 1) + x^2(x + 1)^{2n} - 2x^n(1 + x). \tag{9}$$

Consequently, by equations 5 and 9 have:

$$\begin{aligned}
D(B_{n,5}, x) &= x^3(x+1)^{2n} + 2x^n + (x^2 + 2x)^n(2x^2 + x - 1) \\
&\quad + (x^2 + 2x)^n(2x + 1) + x^2(x+1)^{2n} - 2x^n(1+x) \\
&= x^2(x+1)^{2n+1} - 2x^{n+1} + (x^2 + 2x)^n(2x^2 + 3x). \quad \square
\end{aligned}$$

Figure 10 shows the domination roots of book graphs $B_{n,5}$ for $n \leq 30$.

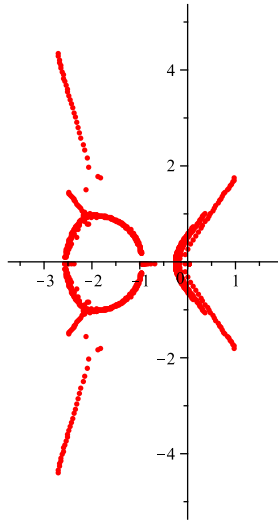


Figure 10: Domination roots of graphs $B_{n,5}$, for $1 \leq n \leq 30$.

3.1 Limits of domination roots of book graphs B_n and $B_{n,5}$.

In this section we consider the complex domination roots of book graphs. The plot in Figures 7 and 10 suggest that the roots tend to lie on a curve. In order to find the limiting curve, we will need a definition and a well known result.

Definition 1 *If $f_n(x)$ is a family of (complex) polynomials, we say that a number $z \in \mathbb{C}$ is a limit of roots of $f_n(x)$ if either $f_n(z) = 0$ for all sufficiently large n or z is a limit point of the set $\mathbb{R}(f_n(x))$, where $\mathbb{R}(f_n(x))$ is the union of the roots of the $f_n(x)$.*

The following restatement of the Beraha-Kahane-Weiss theorem [5] can be found in [6].

Theorem 9. Suppose $f_n(x)$ is a family of polynomials such that

$$f_n(x) = \alpha_1(x)\lambda_1(x)^n + \alpha_2(x)\lambda_2(x)^n + \dots + \alpha_k(x)\lambda_k(x)^n \quad (10)$$

where the $\alpha_i(x)$ and the $\lambda_i(x)$ are fixed non-zero polynomials, such that for no pair $i \neq j$ is $\lambda_i(x) \equiv \omega\lambda_j(x)$ for some $\omega \in \mathbb{C}$ of unit modulus. Then $z \in \mathbb{C}$ is a limit of roots of $f_n(x)$ if and only if either

(i) two or more of the $\lambda_i(z)$ are of equal modulus, and strictly greater (in modulus) than the others; or

(ii) for some j , $\lambda_j(z)$ has modulus strictly greater than all the other $\lambda_i(z)$, and $\alpha_j(z) = 0$.

The following Theorem gives the limits of the domination roots of book graphs B_n .

Theorem 10. The limit of domination roots of book graphs are $x = -\frac{1}{2}$ and $x = 0$ together with the part of the circle $|x + 2| = 1$ with real part at least $-\frac{3}{2} - \frac{\sqrt{2}}{2}$, the portions of the hyperbola $(\Re(x) + 1)^2 - (\Im(x))^2 = \frac{1}{2}$, $\Re(x) \notin [\frac{-3-\sqrt{2}}{2}, \frac{-2-\sqrt{2}}{2}]$, plus the portion of the curve $|x + 1|^2 = |x|$ with real part at most $-\frac{3}{2} - \frac{\sqrt{2}}{2}$.

Proof. By Theorem 5, the domination polynomial of B_n is,

$$\begin{aligned} D(B_n, x) &= (2x + 1)(x^2 + 2x)^n + x^2(x + 1)^{2n} - 2x^n \\ &= \alpha_1(x)\lambda_1^n(x) + \alpha_2(x)\lambda_2^n(x) + \alpha_3(x)\lambda_3^n(x), \end{aligned}$$

where

$$\begin{aligned} \alpha_1(x) &= 2x + 1, \quad \lambda_1(x) = x^2 + 2x, \\ \alpha_2(x) &= x^2, \quad \lambda_2(x) = (x + 1)^2, \end{aligned}$$

and

$$\alpha_3(x) = -2, \quad \lambda_3(x) = x.$$

Clearly α_1 , α_2 and α_3 are not identically zero. Also, no $\lambda_i = \omega\lambda_j$ for $i \neq j$ and a complex number ω of modulus 1. Therefore, the initial conditions of Theorem 9 are satisfied. Now, applying part (i) of Theorem 9, we consider 4 different cases:

$$(i) \quad |\lambda_1| = |\lambda_2| = |\lambda_3|$$

$$(ii) \quad |\lambda_1| = |\lambda_2| > |\lambda_3|$$

$$(iii) \quad |\lambda_1| = |\lambda_3| > |\lambda_2|$$

$$(iv) \quad |\lambda_2| = |\lambda_3| > |\lambda_1|$$

case (i): Assume that $|x^2 + 2x| = |(x+1)^2| = |x|$. Then $|x^2 + 2x| = |x|$ implies that x lies on the unit circle centered -2 ($|x - (-2)| = 1$) and $|x^2 + 2x| = |(x+1)^2|$ by setting $y = x+1$, that is,

$$|y^2 - 1| = |y^2|.$$

To find this curve, let $a = \Re(y)$ and $b = \Im(y)$. Then by substituting in $y = a + ib$ and squaring both sides, we have

$$(a^2 - 1 - b^2)^2 + (2ab)^2 = (a^2 - b^2)^2 + (2ab)^2.$$

This is equivalent to

$$a^2 - b^2 = \frac{1}{2},$$

a hyperbola. Hence, we converting back to variable x , we have the following hyperbola

$$(\Re(x) + 1)^2 - (\Im(x))^2 = \frac{1}{2}.$$

Now suppose that $|(x+1)^2| = |x|$, this curve is semi-cardioid which has shown in Figure 11. Therefore, the two points of intersection, $\frac{-3-\sqrt{2}}{2} \pm \frac{\sqrt{1+2\sqrt{2}}}{2}i$, are limits of roots.

case (ii): Assume that $|x^2 + 2x| = |(x+1)^2| > |x|$. Then $|x^2 + 2x| = |(x+1)^2|$ implies that x lies on the hyperbola $(\Re(x) + 1)^2 - (\Im(x))^2 = \frac{1}{2}$. And $|x^2 + 2x| > |x|$ implies that x lies outside the unit circle centered -2 ($|x - (-2)| = 1$), and $|(x+1)^2| > |x|$ implies that x lies outside the curve $|(x+1)^2| = |x|$. Therefore, the complex numbers x that satisfy

$$(\Re(x) + 1)^2 - (\Im(x))^2 = \frac{1}{2}, \quad \Re(x) \notin \left[\frac{-3-\sqrt{2}}{2}, \frac{-2-\sqrt{2}}{2} \right]$$

are limits of roots.

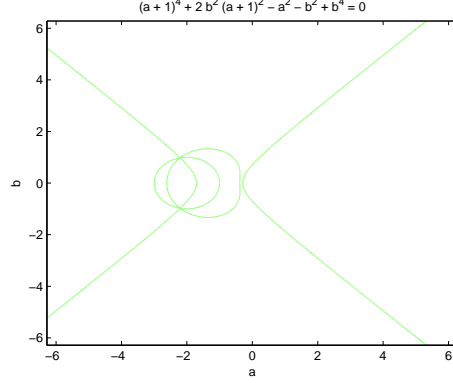


Figure 11: The curves in case (i) in the proof of Theorem .

case (iii): Assume that $|x^2 + 2x| = |x| > |(x+1)^2|$. Then $|x^2 + 2x| = |x|$ implies that x lies on the unit circle centered -2 ($|x - (-2)| = 1$) and $|x^2 + 2x| > |(x+1)^2|$ implies that x satisfy in the following inequality

$$(\Re(x) + 1)^2 - (\Im(x))^2 < \frac{1}{2}.$$

The inequality $|x| > |(x+1)^2|$ implies that x lies inside the curve $|(x+1)^2| = |x|$. Therefore, the complex numbers x that satisfy $|x - (-2)| = 1$ with real part at least $\frac{-3-\sqrt{2}}{2}$ are limits of roots.

case (iv): Assume that $|(x+1)^2| = |x| > |x^2 + 2x|$. As we observed before, the equality $|(x+1)^2| = |x|$ is semi-cardioid which has shown in Figure 11. The inequality $|x| > |x^2 + 2x|$ implies that x lies inside the unit circle centered -2 ($|x - (-2)| = 1$), and $|(x+1)^2| > |x^2 + 2x|$ implies that x satisfy in the following inequality

$$(\Re(x) + 1)^2 - (\Im(x))^2 > \frac{1}{2}.$$

Therefore, the complex numbers x that satisfy on the curve $|(x+1)^2| = |x|$ with real part at most $\frac{-3-\sqrt{2}}{2}$ are limits of roots.

Finally by Part (ii) of Theorem 9, since α_3 is never 0, and $\alpha_2 = 0$ iff $x = 0$, in this case $|\lambda_2(0)| = |1| > 0 = |\lambda_1(0)| = |\lambda_3(0)|$, and $\alpha_1 = 0$ iff $x = -\frac{1}{2}$, and also in this case $|\lambda_1(-\frac{1}{2})| = |-\frac{3}{4}| > \frac{1}{4} = |\lambda_2(-\frac{1}{2})|$ and $|\lambda_1(-\frac{1}{2})| = |-\frac{3}{4}| > \frac{1}{2} = |\lambda_3(-\frac{1}{2})|$, so we conclude $x = 0$ and $x = -\frac{1}{2}$ are limit of domination roots of book graphs.

The union of the curves and points above yield the desired result. \square

Along the same lines, we can show:

Theorem 11. *The limit of roots of the domination polynomial of the book graphs $B_{n,5}$, consist of the part of the circle $|x+2|=1$ with real part at least $-\frac{3}{2}-\frac{\sqrt{2}}{2}$, the portions of the hyperbola $(\Re(x)+1)^2 - (\Im(x))^2 = \frac{1}{2}$, $\Re(x) \notin [\frac{-3-\sqrt{2}}{2}, \frac{-2-\sqrt{2}}{2}]$, plus the portion of the curve $|x+1|^2 = |x|$ with real part at most $-\frac{3}{2}-\frac{\sqrt{2}}{2}$.*

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